Functions of a complex variable

Derivative:

If $w = f(z)$ where z and w are complex numbers, the derivative $\frac{\mathrm{d}w}{\mathrm{d}z}$ at z_0 is

$$f'(z_0) = \lim_{z \to z_0} \left[\frac{f(z) - f(z_0)}{z - z_0} \right]$$

provided that the limit exists as $z \to z_0$ along *any* path. If $f(z)$ has a derivative at a point z_0 and at all points in some neighbourhood of z_0 then $f(z)$ is said to be **analytic** at z_0. If $f(z)$ is analytic at all points in an (open) region R then $f(z)$ is said to be **analytic** in R.

Cauchy-Riemann equations:

If $z = x + jy$ and $w = f(z) = u(x, y) + jv(x, y)$ where x, y, u and v are real variables, and $f(z)$ is analytic in some region R of the z plane, then the **Cauchy-Riemann equations** hold throughout R:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

If these partial derivatives are continuous within R, the Cauchy-Riemann equations are sufficient conditions to ensure $f(z)$ is analytic. Furthermore,

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

Singularities:

If $f(z)$ fails to be analytic at a point z_0 but is analytic at some point in every neighbourhood of z_0 then z_0 is called a **singular point** of $f(z)$.

Laurent series:

If $f(z)$ is analytic on concentric circles C_1 and C_2 of radii r_1 and r_2, centred at z_0, and also analytic throughout the annular region between the circles, then for each point z within the annulus, $f(z)$ may be represented by the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

in which c_n are complex constants. The series may be written

$$f(z) = \sum_{n=-\infty}^{-1} c_n (z - z_0)^n + \sum_{n=0}^{\infty} c_n (z - z_0)^n.$$

Poles:

The first sum on the right is the **principal part**. If there are only a finite number of terms in the principal part e.g.

$$\begin{aligned} f(z) &= \frac{c_{-m}}{(z-z_0)^m} + \ldots + \frac{c_{-1}}{(z-z_0)} \\ &\quad + c_0 + c_1(z-z_0) + \ldots + c_m(z-z_0)^m + \ldots \end{aligned}$$

in which $c_{-m} \neq 0$, then $f(z)$ has a singularity called a **pole of order** m at $z = z_0$. A pole of order 1 is called a **simple pole**. If there are infinitely many terms in the principal part, z_0 is called an **isolated essential singularity**. If the principal part is zero, then $f(z)$ has a **removable singularity** at $z = z_0$ and the Laurent series reduces to a Taylor series.

Residues:

If $f(z)$ has a pole at $z = z_0$ then the coefficient, c_{-1}, of $\frac{1}{z-z_0}$ in the Laurent expansion is called the **residue** of $f(z)$ at $z = z_0$. The residue at a pole of order m is given by:

$$\frac{1}{(m-1)!} \lim_{z \to z_0} \left\{ \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} \left[(z-z_0)^m f(z) \right] \right\}.$$