Fourier Series

Fourier Series:

If $f(t)$ is periodic with period T its Fourier series is:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2n\pi t}{T} + b_n \sin \frac{2n\pi t}{T} \right)$$

or equivalently, if $\omega = 2\pi/T$,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t).$$

a_n and b_n are the **Fourier coefficients:**

$$a_n = \frac{2}{T} \int_d^{d+T} f(t) \cos \frac{2n\pi t}{T} \mathrm{d}t, \quad \text{for } n = 0, 1, 2, 3 \ldots$$

$$b_n = \frac{2}{T} \int_d^{d+T} f(t) \sin \frac{2n\pi t}{T} \mathrm{d}t, \quad \text{for } n = 1, 2, 3 \ldots$$

where d can be chosen to have any value.

If $f(t)$ is odd, $a_n \equiv 0$ and $f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega t$.

If $f(t)$ is even, $b_n \equiv 0$ and $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t$.

Parseval's theorem:

$$\frac{2}{T} \int_0^T (f(t))^2 \mathrm{d}t = \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

Complex form:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{j2n\pi t/T},$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\mathrm{e}^{-j2n\pi t/T}\mathrm{d}t.$$

Half-range sine series:

Given $f(t)$ for $0 < t < \frac{T}{2}$, its odd periodic extension has period T and Fourier series given by

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi t}{T}.$$

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin\frac{2n\pi t}{T}\mathrm{d}t \qquad \text{for } n = 1, 2, 3\ldots$$

Half-range cosine series:

Given $f(t)$ for $0 < t < \frac{T}{2}$, its even periodic extension has period T and Fourier series given by

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi t}{T}.$$

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos\frac{2n\pi t}{T}\mathrm{d}t \qquad \text{for } n = 0, 1, 2, 3\ldots$$