Vector Calculus

$$\text{grad} \equiv \nabla \qquad \text{div} \equiv \nabla\, \cdot \qquad \text{curl} \equiv \nabla \times$$

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

$$\text{Laplacian} \equiv \nabla^2 \equiv \text{div(grad)} \equiv \nabla \cdot \nabla \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

If $\Phi(x, y, z)$ is a scalar field and $\mathbf{v}(x, y, z) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is a vector field

$$\text{grad}\,\Phi = \nabla\Phi = \frac{\partial \Phi}{\partial x}\mathbf{i} + \frac{\partial \Phi}{\partial y}\mathbf{j} + \frac{\partial \Phi}{\partial z}\mathbf{k} \qquad \text{a vector.}$$

$$\text{div}\,\mathbf{v} = \nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \qquad \text{a scalar.}$$

$$\text{curl}\,\mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} \qquad \text{a vector.}$$

$$\nabla^2\Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}.$$

$$\nabla^2\mathbf{v} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}).$$

Vector calculus identities:

$\text{grad}(\Phi\psi) = \Phi\,\text{grad}\,\psi + \psi\,\text{grad}\,\Phi$

$\text{div}\,(\Phi\mathbf{a}) = \Phi\,\text{div}\,\mathbf{a} + \mathbf{a}\cdot\text{grad}\,\Phi$

$\text{curl}\,(\Phi\mathbf{a}) = \Phi\,\text{curl}\,\mathbf{a} + \text{grad}\,\Phi \times \mathbf{a}$

$\text{curl}\,\text{grad}\,\Phi = \mathbf{0}, \qquad \text{div}\,\text{curl}\,\mathbf{a} = 0$

$\text{curl}\,\text{curl}\,\mathbf{a} = \text{grad}\,\text{div}\,\mathbf{a} - \nabla^2\mathbf{a}$

$\text{grad}(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{b}\cdot\text{grad})\mathbf{a} + (\mathbf{a}\cdot\text{grad})\,\mathbf{b} + \mathbf{b}\times\text{curl}\,\mathbf{a} + \mathbf{a}\times\text{curl}\,\mathbf{b}$

$\text{div}(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot\text{curl}\,\mathbf{a} - \mathbf{a}\cdot\text{curl}\,\mathbf{b}$

$\text{curl}(\mathbf{a}\times\mathbf{b}) = (\mathbf{b}\cdot\text{grad})\mathbf{a} - (\mathbf{a}\cdot\text{grad})\mathbf{b} + \mathbf{a}\,\text{div}\,\mathbf{b} - \mathbf{b}\,\text{div}\,\mathbf{a}$

Green's theorem in the plane:

$$\oint_C (P\mathrm{d}x + Q\mathrm{d}y) = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y.$$

Stokes' theorem:

$$\oint_C \mathbf{v}\cdot\mathrm{d}\mathbf{r} = \int_S \mathrm{curl}\,\mathbf{v}\cdot\mathrm{d}\mathbf{S}.$$

The divergence theorem:

$$\oint_S \mathbf{v}\cdot\mathrm{d}\mathbf{S} = \int_V \mathrm{div}\,\mathbf{v}\,\mathrm{d}V.$$