mathcentre

Using a table of derivatives

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In this unit we construct a Table of Derivatives of commonly occurring functions. This is done using the knowledge gained in previous units on differentiation from first principles.

Rules, known as linearity rules, for constant multiples of functions, and for the sum/difference of two functions are also given and illustrated with examples.

Finally, the table is extended further by making use of the chain rule for differentiating a function of a function.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- construct and use a table of derivatives of commonly occurring functions

Contents

1. Introduction

Other units have explained how to differentiate all the common functions, such as x^n, $\cos x$, $\sin x$, e^x, $\ln x$ and so on. Usually this has been done from first principles. In this unit we pull all these results together and construct a table of standard derivatives which you can consult as the need arises.

2. A Table of Derivatives

Commonly occurring functions and their derivatives are given in the Table below.

function $f(x)$	derivative $\dfrac{df}{dx}$ or $f'(x)$	
c	0	c is any constant
x	1	
$2x$	2	
x^n	nx^{n-1}	n is any real number
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	
e^x	e^x	
$\ln x$	$\dfrac{1}{x}$	

We can make this table more useful by extending the range of functions it includes. We can do this using rules, known as **linearity rules**: these are the **constant multiplier rule** and the **addition rule**.

3. The constant multiplier rule

Suppose we have a function $f(x)$ and multiply it by a constant, c say. The derivative of this is given by the following rule:

$$\frac{d}{dx}(c\,f(x)) = c\,\frac{df}{dx}$$

In other words, we simply differentiate the function and multiply the result by the constant c.

Examples

If $y = 2\sin x$ then $\dfrac{dy}{dx} = 2\dfrac{d}{dx}(\sin x) = 2\cos x$.

If $y = -5\sin x$ then $\dfrac{dy}{dx} = -5\dfrac{d}{dx}(\sin x) = -5\cos x$.

Proof from first principles of the constant multiplier rule

Consider the function $g(x) = cf(x)$ where c is a constant.
Using the definition of the derivative of $g(x)$ we have

$$\begin{aligned} g'(x) &= \lim_{\delta x \to 0} \frac{g(x+\delta x) - g(x)}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{c\,f(x+\delta x) - c\,f(x)}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{c(f(x+\delta x) - f(x))}{\delta x} \end{aligned}$$

The constant c is unaffected by the limiting process and so can be taken outside the limit:

$$g'(x) = c \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

But the term following the c on the right hand side is just the definition of the derivative of $f(x)$.
So we have

$$g'(x) = c\,\frac{df}{dx} = c\,f'(x)$$

Key Point

The constant multiplier rule:

$$\frac{d}{dx}\,(c\,f(x)) = c\,\frac{df}{dx}$$

Exercise 1

Find the derivative of each of the following:

a)	$5x^4$	b)	$12x$	c)	$4x^{-2}$	d)	$8\cos x$	e)	$-3\cos x$
f)	$2e^x$	g)	$3\ln x$	h)	$-7e^x$	i)	$-2\sin x$	j)	$-4\ln x$

4. The addition and subtraction rules

The first of these rules enables us to differentiate the sum of two functions, e.g. $f(x) + g(x)$. The rule states that to differentiate this sum we simply differentiate each term separately and then add the results:

$$\frac{d}{dx}\,(f(x) + g(x)) = \frac{df}{dx} + \frac{dg}{dx}$$

Similarly, if we have the difference of two functions:

$$\frac{d}{dx}\left(f(x) - g(x)\right) = \frac{df}{dx} - \frac{dg}{dx}$$

Key Point

Sum and difference rules:

$$\frac{d}{dx}\left(f(x) \pm g(x)\right) = \frac{df}{dx} \pm \frac{dg}{dx}$$

These rules can be added to the Table given on Page 2.

Example

Suppose we wish to differentiate $y = 2x^3 - 6\cos x$.

We differentiate each term separately, and make use of the constant multiplier rule:

$$\begin{aligned} \frac{d}{dx}\left(2x^3 - 6\cos x\right) &= \frac{d}{dx}(2x^3) - \frac{d}{dx}(6\cos x) \\ &= 2\frac{d}{dx}(x^3) - 6\frac{d}{dx}(\cos x) \\ &= 2(3x^2) - 6(-\sin x) \\ &= 6x^2 + 6\sin x \end{aligned}$$

5. Further extensions to the Table

In this section we extend the Table by looking at functions of the form $\sin mx$, $\cos mx$, e^{mx} and $\ln mx$.

Example

Suppose we wish to differentiate $y = \sin mx$ in order to find $\dfrac{dy}{dx}$.

We begin by making the substitution $u = mx$. This simplifies the original function to give $y = \sin u$.

To find $\dfrac{dy}{dx}$ we use a rule (called the chain rule, or the function of a function rule) which states

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

This rule is dealt with at length in another unit. For now we will simply quote and use it.

We need to calculate $\dfrac{dy}{du}$: since $y = \sin u$ it follows that $\dfrac{dy}{du} = \cos u$.

We also need to calculate $\dfrac{du}{dx}$: since $u = mx$ it follows that $\dfrac{du}{dx} = m$ because m is a constant.

Substituting into the chain rule we find

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\times\frac{du}{dx}\\ &= \cos u \times m\\ &= m\cos u\\ &= m\cos mx\end{aligned}$$

since $u = mx$.

We have shown that if $y = \sin mx$ then $\dfrac{dy}{dx} = m\cos mx$.

In a similar way it is straightforward to show that if $y = \cos mx$ then $\dfrac{dy}{dx} = -m\sin mx$.

Example

Suppose we wish to differentiate $y = \mathrm{e}^{mx}$.

Again, we substitute $u = mx$ so that $y = \mathrm{e}^u$. We then use the chain rule:

We need to calculate $\dfrac{dy}{du}$: since $y = \mathrm{e}^u$ it follows that $\dfrac{dy}{du} = \mathrm{e}^u$.

We also need to calculate $\dfrac{du}{dx}$: since $u = mx$ it follows that $\dfrac{du}{dx} = m$ because m is a constant.

Substituting into the chain rule we find

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\times\frac{du}{dx}\\ &= \mathrm{e}^u \times m\\ &= m\mathrm{e}^u\\ &= m\mathrm{e}^{mx}\end{aligned}$$

since $u = mx$.

We have shown that if $y = \mathrm{e}^{mx}$ then $\dfrac{dy}{dx} = m\mathrm{e}^{mx}$.

Example

Suppose we wish to differentiate $y = \ln mx$.

Again, we substitute $u = mx$ so that $y = \ln u$. We then use the chain rule:

We need to calculate $\dfrac{dy}{du}$: since $y = \ln u$ it follows that $\dfrac{dy}{du} = \dfrac{1}{u}$.

We also need to calculate $\dfrac{du}{dx}$: since $u = mx$ it follows that $\dfrac{du}{dx} = m$ because m is a constant.

Substituting into the chain rule we find

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\times\frac{du}{dx}\\ &= \frac{1}{u}\times m\\ &= m\frac{1}{mx}\\ &= \frac{1}{x}\end{aligned}$$

We have shown that if $y = \ln mx$ for constant m, then $\dfrac{dy}{dx} = \dfrac{1}{x}$.

This result could also have been obtained by using the laws of logarithms to rewrite $y = \ln mx$ as $y = \ln m + \ln x$. Then we could differentiate this sum, term by term. The first term has derivative zero. This is because the the logarithm of a constant is still a constant and so its derivative is zero. The derivative of the second term is simply $\frac{1}{x}$.

Example

Suppose we wish to differentiate $y = \ln(ax + b)$ where a and b are constants.

This time we substitute $u = ax + b$ so that $y = \ln u$. We then use the chain rule:
We need to calculate $\dfrac{dy}{du}$: since $y = \ln u$ it follows that $\dfrac{dy}{du} = \dfrac{1}{u}$.
We also need to calculate $\dfrac{du}{dx}$: since $u = ax + b$ it follows that $\dfrac{du}{dx} = a$ because a and b are constants.

Substituting into the chain rule we find

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \frac{1}{u} \times a \\ &= \frac{a}{u} \\ &= \frac{a}{ax+b}\end{aligned}$$

since $u = ax + b$.
We have shown that if $y = \ln(ax + b)$ for constants a and b, then $\dfrac{dy}{dx} = \dfrac{a}{ax+b}$.

The results we have generated in the preceding sections can be added to the table of derivatives given on page 2 to produce a more complete and thereby more useful table:

function $f(x)$	derivative $\dfrac{df}{dx}$ or $f'(x)$	
c	0	c is any constant
x	1	
$2x$	2	
x^n	nx^{n-1}	n is any real number
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	
e^x	e^x	
$\ln x$	$\dfrac{1}{x}$	
$cf(x)$	$cf'(x)$	c constant
$f(x) \pm g(x)$	$f'(x) \pm g'(x)$	
$\sin mx$	$m\cos mx$	m is a constant
$\cos mx$	$-m\sin mx$	m is a constant
e^{mx}	$m\mathrm{e}^{mx}$	m is a constant
$\ln mx$	$\frac{1}{x}$	
$\ln(ax+b)$	$\frac{a}{ax+b}$	

Exercise 2
Find the derivative of each of the following:

a) $\sin 4x$	b) e^{5x}	c) $\cos 3x$	d) $\ln 5x$
e) $2\sin 3x$	f) $4e^{-2x}$	g) $4\cos(-2x)$	h) $\ln(3x+2)$
i) $4\sin 2x + 3\cos 3x$	j) $e^{2x} - e^{-2x}$	k) $\ln 2x + 2\sin 3x$	l) $4\cos x - 2\ln(x+4)$

Answers

Exercise 1

a) $20x^3$ b) 12 c) $-8x^{-3}$ d) $-8\sin x$ e) $3\sin x$
f) $2e^x$ g) $\dfrac{3}{x}$ h) $-7e^x$ i) $-2\cos x$ j) $-\dfrac{4}{x}$

Exercise 2

a) $4\cos 4x$ b) $5e^{5x}$ c) $-3\sin 3x$ d) $\dfrac{1}{x}$ e) $6\cos 3x$ f) $-8e^{-2x}$ g) $8\sin(-2x)$
h) $\dfrac{3}{3x+2}$ i) $8\cos 2x - 9\sin 3x$ j) $2e^{2x} + 2e^{-2x}$ k) $\dfrac{1}{x} + 6\cos 3x$ l) $-4\sin x - \dfrac{2}{x+4}$

