Composition of functions

mc-TY-composite-2009-1

We can build up complicated functions from simple functions by using the process of composition, where the output of one function becomes the input of another. It is also sometimes necessary to carry out the reverse process, decomposing a complicated function into two or more simple functions.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- write down both the composite functions gf and fg given two suitable functions f and g,
- write a complicated function as a composition gf,
- determine whether two given functions f and g are suitable for composition,
- find the domain and range of a composite function gf given the functions f and g.

Contents

1. Introduction

The composition of two functions g and f is the new function we get by performing f first, and then performing g. For example, if we let f be the function given by $f(x) = x^2$ and let g be the function given by $g(x) = x + 3$, then the composition of g with f is called gf and is worked out as

$$gf(x) = g(f(x)) .$$

So we write down what $f(x)$ is first, and then we apply g to the whole of $f(x)$. In this case, if we apply g to something we add 3 to it. So if we apply g to x^2, we add three to x^2. So we obtain

$$gf(x) = g(f(x)) = g(x^2) = x^2 + 3 .$$

Here is another example of composition of functions. This time let f be the function given by $f(x) = 2x$ and let g be the function given by $g(x) = e^x$. As before, we write down $f(x)$ first, and then apply g to the whole of $f(x)$. In this case, $f(x)$ is just $2x$. Applying the function g then raises e to the power $f(x)$. So we obtain

$$gf(x) = g(f(x)) = g(2x) = \mathrm{e}^{2x} .$$

Sometimes the composition of two functions is called a 'function of a function', and sometimes gf is written $g \circ f$. You don't have to use this notation yourself, but it is a good idea to remember what it means because you might see it used in textbooks.

Key Point

A composed function gf is the function given by

$$gf(x) = g(f(x)).$$

This is sometimes called a function of a function. An alternative notation for gf is $g \circ f$.

Exercises

1. Work out $gf(x)$ for the following pairs of functions:

(a) $f(x) = 3x$, $g(x) = 2x^2 - 5$, (b) $f(x) = e^{4x}$, $g(x) = \sqrt{x}$,
(c) $f(x) = \sin x$, $g(x) = 1/x$.

2. Order of composition

The order in which we compose functions makes a big difference to the result. You can see this if we change the order of the functions in the first example. We have taken $f(x) = x^2$ and $g(x) = x + 3$. Then $fg(x)$ is given by taking $g(x)$, which is $x + 3$, and applying f to all of it. This gives us

$$fg(x) = f(x+3) = (x+3)^2 = x^2 + 6x + 9\,.$$

You can see that this is not the same as $gf(x)$, because

$$gf(x) = x^2 + 3$$

and this does not in general equal $x^2 + 6x + 9$.

Key Point

In general $gf(x)$ is not equal to $fg(x)$.

Exercises

2. Work out $fg(x)$ for the following pairs of functions, and compare the results to those you obtained for Exercise 1:

(a) $f(x) = 3x$, $g(x) = 2x^2 - 5$, (b) $f(x) = e^{4x}$, $g(x) = \sqrt{x}$,
(c) $f(x) = \sin x$, $g(x) = 1/x$.

3. Decomposition of a function

Sometimes we can write a given function as the composition of two other functions. This is called decomposing the function. For example, take the function, $h(x) = e^{2x}$. We have already seen that this function may be written as a composite function $h(x) = gf(x)$, where

$$g(x) = e^x, \qquad f(x) = 2x\,.$$

Let us take another example. Suppose we have been given the function

$$h(x) = \sqrt[3]{2x+1}\,.$$

Here we see again that the function can be performed in two stages. We take x, and we first apply the function $f(x) = 2x + 1$. Then we take the cube root of the result. So if we let $g(x) = \sqrt[3]{x}$ then

$$h(x) = \sqrt[3]{2x+1} = g(2x+1) = g(f(x))\,.$$

It is important to be able to decompose functions in later work in the calculus.

Key Point

Sometimes we can write a function as the composition of two other functions. This process is called decomposing the function.

Exercises

3. Decompose the following functions into the form gf:

(a) $6x+3$, (b) $4x^2$, (c) $4x^2$ (in a different way), (d) e^{x+4}, (e) x^2+2x+1.

4. Domains and ranges of composed functions

Sometimes you will meet pairs of functions that cannot be composed. For example, take the two functions $f(x) = -x^2$ and $g(x) = \ln(x)$. Then we know that a square is always positive or zero, so $f(x) \le 0$ for any x. But we also know that the natural logarithm $\ln x$ is defined only for positive numbers. So in this case, $g(f(x))$ is not defined for any values of x. The composed function gf does not exist for these two functions g and f.

There are also functions that cannot be composed for every x, but that can be composed if we restrict the values of x. For example, let us take

$$f(x) = 4x - 6, \qquad g(x) = \sqrt{x}\,.$$

Now only positive numbers, or zero, have real square roots. So g is defined only for numbers greater than or equal to zero. Therefore $g(f(x))$ can have a value only if $f(x)$ is greater than or equal to zero. You can work out that

$$f(x) \ge 0 \qquad \text{only when} \qquad x \ge \tfrac{3}{2}\,.$$

So the composed function $gf(x)$ can be defined only for $x \ge \frac{3}{2}$, and therefore the domain of the function gf is $x \ge \frac{3}{2}$. In general, the domain of a composed function is either the same as the domain of the first function, or else lies inside it. If x is a valid input for the composed function gf then it must be a valid input for the individual function f.

The range of a function is the set of all values a function can take. For example, the range of the function $f(x) = e^x$ is given by $f(x) > 0$, because e^x is always greater than zero. As another example, if $f(x) = \sin x$ then the range is given by $-1 \le f(x) \le 1$.

If we have a composed function gf then its range must lie within the range of the second function g. Here is an example to show this. Take

$$f(x) = x - 8, \qquad g(x) = x^2\,.$$

Now the first function $f(x)$ can take any value. But the second function $g(x)$ sends any number to its square, which is greater than or equal to zero. So

$$gf(x) = g(x-8) = (x-8)^2 = x^2 - 16x + 64 \geq 0\,.$$

Whatever $f(x)$ is, $g(f(x))$ must be greater than or equal to zero, because g applied to anything is greater than or equal to zero. So the composed function $gf(x) = x^2 - 16x + 64$ can take only values that are greater than or equal to zero.

What happens if we compose the functions the other way round? We shall take g first, and then f. So

$$f(g(x)) = f(x^2) = x^2 - 8 \geq -8\,.$$

The range of fg is given by $f(g(x)) \geq -8$. So in this case, the range of the composed function $f(g(x))$ is contained in the range of f, but it is not the whole of the range of f. And in general, the range of a composed function is either the same as the range of the second function, or else lies inside it. If a value is a possible output from a composed function then it must be a possible output from the second function.

Key Point

Some pairs of functions cannot be composed. Some pairs of functions can be composed only for certain values of x.

The domain of a composed function is either the same as the domain of the first function, or else lies inside it.

The range of a composed function is either the same as the range of the second function, or else lies inside it.

Exercises

4. For the following pairs of functions, find the domain and range of the composed function gf:

(a) $f(x) = 2x,\ g(x) = \sin x$, (b) $f(x) = x^2,\ g(x) = e^x$
(c) $f(x) = -x,\ g(x) = \ln x$, (d) $f(x) = 1/x,\ g(x) = 2\sin x$.

Answers

1. (a) $18x^2 - 5$ (b) e^{2x} (c) $\operatorname{cosec} x$ (or $1/\sin x$)

2. (a) $6x^2 - 15$ (b) $e^{4\sqrt{x}}$ (c) $\sin(1/x)$

3. (a) $f(x) = 6x,\ g(x) = x + 3$ (b) $f(x) = x^2,\ g(x) = 4x$ (c) $f(x) = 2x,\ g(x) = x^2$
(d) $f(x) = x + 4,\ g(x) = e^x$ (e) $f(x) = x + 1,\ g(x) = x^2$

4. (a) domain is all real x, range is $-1 \leq y \leq 1$
(b) domain is all real x, range is $y \geq 1$
(c) domain is $x < 0$, range is all real y
(d) domain is $x \neq 0$, range is $-2 \leq y \leq 2$

